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Multiple harmonic oscillator zeta functions

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Abstract. The partition function for a quantum mechanical system can—at least formally—be expressed as a high-temperature series in terms of the ζ function of the system's Hamiltonian. This is done explicitly for a system of non-interacting harmonic oscillators. The Hamilton ζ function belongs to a class of ζ functions $\Sigma f(n_i)(n_1\omega_1+n_2\omega_2+\ldots+n_N\omega_N+a)^{-s}$ summed over non-negative integers n_i. Very little is known about these ζ functions, and an investigation of some of their properties is presented.

1. Introduction

A simple argument given in § 2 suggests that the partition function Z of a quantum mechanical system with Hamiltonian H has a high-T expansion (T = temperature) in terms of the ζ function $\zeta(s|H)$ of H. In § 2 this will be shown to be true for a system of non-interacting harmonic oscillators. A comparable result is obtained for a system of non-interacting supersymmetric oscillators.

The Hamiltonian ζ function for N non-interacting oscillators:

$$\zeta(s|H)=\sum_{n_i=0}^{\infty}(n_1\omega_1+n_2\omega_2+\ldots+n_N\omega_N+\alpha)^{-s} \tag{1.1}$$

is distinguished from the operator ζ functions usually encountered in field theory in that the denominator function is *linear* rather than quadratic in the summation indices n_i. Whereas ζ functions of the latter type have often been studied, there seems to be virtually nothing in the literature on ζ functions of the form (1.1). Therefore, a substantial part of the present paper (§ 3) will be devoted to these ζ functions and their properties. This exercise brings into focus important differences between single-sum and multiple-sum ζ functions which have not previously received attention.

2. Hamiltonian zeta function and the partition function

2.1. *High-T expansion of the partition function*

Consider the partition function for a quantum mechanical system with discrete energy levels $E_n>0$:

$$Z=\operatorname{Tr}\exp(-\beta H)=\sum_{n=0}^{\infty}\exp(-\beta E_n) \tag{2.1}$$

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where as always a sum over states is understood. By assumption the Hamiltonian H is a well behaved operator whose zeta function

$$\zeta(s|H) \equiv \sum_{n=0}^{\infty} 1/E_n^s \tag{2.2}$$

is meromorphic in s with poles along the real axis, ending with a rightmost pole at finite position $s = C > 0$ [1]. The series (2.2) converges absolutely for $\operatorname{Re} s > C$; however, the function defined by this series can be continued to all s and is meromorphic as stated.

An interesting possibility is to express Z as a high-T series in terms of $\zeta(s|H)$. Formally this can be done in the following way. Expand the exponential in (2.1) to obtain the double series

$$Z = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!}(-\beta)^k E_n^k.$$

This step is well defined since the exponential series has an infinite radius of convergence. Next, commute Σ_n with Σ_k and use (2.2) to find

$$Z = \sum_{k=0}^{\infty} \frac{1}{k!}(-\beta)^k \zeta(-k|H) + \{\ \} \tag{2.3}$$

where $\zeta(-k|H)$ is finite as long as $s = -k$ is not a pole of the ζ function. This is the crucial step and it must be carefully justified. The curly bracket $\{\ \}$ represents (unknown) extra terms which may be generated by the commutation of Σ_n with Σ_k. Given $\zeta(s|H)$ as a function of s, it should be possible to justify this commutation and compute $\{\ \}$. Until this has been done, the series (2.3) remains formal and incomplete. The virtue of this expression is that it shifts the burden of computation from direct evaluation of (2.1) to the evaluation of (2.2), which might be less difficult.

In this section we try out the program just described on the exactly solvable harmonic oscillator problem. For a single oscillator, the ζ function (2.2) is just the elementary Hurwitz ζ function $\zeta(s, a)$. When N non-interacting oscillators are considered, a new kind of ζ function (1.1) is encountered whose properties will be investigated in § 3. Supersymmetric oscillators will also be considered. An appropriate ζ function for fermionic harmonic oscillators is identified. The application of the method to less trivial physical systems appears to hold promise, and work in this direction is in progress.

2.2. *Ordinary oscillators*

The energy levels $E_n = (n+a)\omega$ of the harmonic oscillator define the Hamiltonian ζ function:

$$\zeta(s|H) = \sum_{n=0}^{\infty} (n+a)^{-s}\omega^{-s} = \omega^{-s}\zeta(s, a) \tag{2.4}$$

where $\zeta(s, a)$ is the Hurwitz ζ function (see (3.3) below). One readily writes down (2.3) for this case:

$$\begin{aligned} Z_b(\omega) &= \exp(-a\beta\omega)[1-\exp(-\beta\omega)]^{-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}(-\beta\omega)^k \zeta(-k, a) + \left\{\frac{1}{\beta\omega}\right\} \end{aligned} \tag{2.5}$$

where the curly bracket term is straightforward to evaluate [2, 3] using the known meromorphic properties of $\zeta(s, a)$, namely a single pole as $s=1$ with unit residue. Because $\zeta(-k, a) = -B_{k+1}(a)/(k+1)$, $B_{k+1}(1/2) = (2^{-k}-1)B_{k+1}$, and $B_{3,5,\ldots}=0$, where $B_n(a)$ and B_n are Bernoulli polynomials and numbers respectively [4, 5], (2.5) with $a=\frac{1}{2}$ is just the familiar result

$$Z_{\mathrm{b}}(\omega) = (2\sinh \beta\omega/2)^{-1}$$
$$= (\beta\omega)^{-1} + \sum_{k=1,3,5,\ldots} \frac{1}{(k+1)!}\left(\frac{\beta\omega}{2}\right)^{k}(1-2^{k})B_{k+1}. \tag{2.6}$$

For N uncoupled oscillators the partition function becomes

$$Z_{\mathrm{B}} = Z_{\mathrm{b}}(\omega_1)\ldots Z_{\mathrm{b}}(\omega_N)$$
$$= \sum_{n_i=0}^{\infty} \exp[-(n_1+a)\beta\omega_1]\ldots\exp[-(n_N+a)\beta\omega_N]. \tag{2.7}$$

Here we can expand the exponentials individually to find Z_{B} as a product of the high-T series (2.5). Alternatively, we can write the entire summand as a single exponential as in (2.3), and expand this to obtain the explicit (i.e. not in product form) high-T series for Z_{B}. The ζ functions (1.1) are the Hamiltonian ζ functions appearing in this series. To perform this rearrangment to all orders (from a product of infinite series to the explicit high-T series) is not trivial. Nevertheless, the factors $Z_{\mathrm{b}}(\omega)$ in (2.7) are elementary functions and one has no real need for the high-T series of Z_{B}. Therefore, we shall not discuss this series.

2.3. *Supersymmetric oscillators*

Proceeding to supersymmetric oscillators, we find that supersymmetry precisely corresponds to an intrinsic and important property of ζ functions. This property is the pairing of ζ functions with non-alternating and alternating sign under an otherwise identical sum. Such pairs of ζ functions have quite different—yet closely related—properties. The paired Hurwitz ζ functions $\zeta(s, a)$ and $\eta(s, a)$ defined in (3.3) are the ζ functions relevant for bosonic and fermionic oscillators respectively. Little tabulated information is available on $\eta(s, a)$, but in [3] many of the properties of this function were derived, and its close parallel with $\zeta(s, a)$ exhibited.

The partition function for the fermionic sector of the supersymmetric oscillator can be found in a variety of ways. A fermionic oscillator has two energy levels $E'_n = (n-\frac{1}{2})\omega$, $n=0$ or 1 (unoccupied or occupied) [6] and partition function

$$Z_{\mathrm{f}}(\omega) = \exp(-\beta E'_1) + \exp(-\beta E'_2) = 2\cosh(\beta\omega/2). \tag{2.8}$$

A more general approach is to use the path integral formalism. Thus for a fermionic oscillator:

$$Z_{\mathrm{f}}(\omega) = \det{}_{-}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x_0^2}+\omega^2\right)$$
$$= \prod_n \left((2n+1)^2\frac{\pi^2}{\beta^2}+\omega^2\right) = 2\cosh\frac{\beta\omega}{2}$$

where the subscript 'minus' means that antiperiodic boundary conditions around the time circle are used rather than the periodic boundary conditions appropriate for the bosonic sector. This makes the path integral variables Grassmannian and puts the determinant in the numerator. The supersymmetric oscillator partition function is [6] $Z_b Z_f = \coth \beta\omega/2$.

Because only a finite number of fermionic energy levels exist, it would seem impossible to define a 'fermionic' ζ function analogous to $\zeta(s|H)$ for the high-T expansion of Z_f. However, for harmonic oscillators at least, the following definition is the correct one:

$$\eta(s|H) \equiv \sum_{n=0}^{\infty} (-1)^n 1/E_n^s. \tag{2.9}$$

In this definition the *bosonic* energy levels E_n are used, but an alternating sign has been introduced. Thus $\eta(s|H) = \omega^{-s}\eta(s, a)$ for the oscillator. Equation (2.9) is to be used in conjunction with a second definition, giving $1/Z_f$ in terms of the bosonic energy levels:

$$\begin{aligned} 1/Z_f(\omega) &\equiv \sum_{n=0}^{\infty} (-1)^n \exp(-\beta E_n\} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}(-\beta\omega)^k \eta(-k, a) \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{k!}(-\beta\omega)^k E_k(a) \\ &= \exp(-a\beta\omega)[1+\exp(-\beta\omega)]^{-1}. \end{aligned} \tag{2.10}$$

Here there is no extra term and the $E_k(a) = 2\eta(-k, a)$ [3] are Euler polynomials [4].

Equation (2.10) coincides with (2.8) for $a = \frac{1}{2}$ and is therefore correct for the fermionic oscillator. This formula seeks to define the fermionic partition function directly in terms of bosonic energy levels. Its relevance for more complex systems is uncertain at this point. Still, this expression may deserve further study.

For N uncoupled supersymmetric oscillators the inverse fermionic partition function $1/Z_F$ is a product of the functions (2.10). The extension of the fermionic ζ function (2.9) to N oscillators is

$$\eta(s|H) \equiv \sum_{n_i=0}^{\infty} (-1)^{n_1+\ldots+n_N}(n_1\omega_1 + \ldots + n_N\omega_N + \alpha)^{-s} \tag{2.11}$$

with $\alpha = (\omega_1 + \omega_2 + \ldots \omega_N)a$. This ζ function appears in the high-T expansion of $1/Z_F$. Its properties are investigated below.

3. Multiple harmonic oscillator ζ functions

The general form characteristic of multiple harmonic oscillator ζ functions is

$$\sum_{n_i=0}^{\infty} (a_1 n_1 + \ldots + a_N n_N + b)^{-s} f(n_i) \tag{3.1}$$

where the main factor in the summand is linear in the indices n_i and $f(n_i)$ is some chosen function of the n_i. Aside from a formula for $N = 2$ obtained in [7], the author is aware of no other results on these ζ functions. The purpose of this section is to derive some of their properties. It may be helpful to begin with a few general remarks.

3.1. *Single-sum ζ functions*

Consider a generalised ζ function $\zeta(s|x_i)$ defined by an infinite numerical series:

$$\zeta(s|x_i) \equiv \sum_m f_m(s, x_i) \qquad \mathrm{Re}\, s > C \tag{3.2}$$

where the summand $f_m(s, x_i)$ depends on the complex variable s and some parameters x_i. This is a generalisation of the defining equations of the Riemann and Hurwitz ζ functions [5, 8]:

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s} \qquad \eta(s) = \sum_{m=1}^{\infty} (-)^{m+1} m^{-s}$$

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s} \qquad \eta(s, a) = \sum_{n=0}^{\infty} (-)^n (n+a)^{-s}. \tag{3.3}$$

The series for $\zeta(s)$ and $\zeta(s, a)$ converge only for $\mathrm{Re}\, s > 1$. By analytic continuation, these series are shown to define meromorphic functions having poles at $s = 1$ with unit residue. $\eta(s)$ and $\eta(s, a)$ are regular for all finite s. Equation (3.2) is to be understood in the same way—analytic continuation reveals that the numerical series defines a meromorphic function of s. A convenient way of performing this analytic continuation is to derive a regularisation formula for $\zeta(s|x_i)$. The idea is very simple.

A basic observation made in [9] is that many generalised ζ functions and Dirichlet series can be expressed in terms of simple ζ functions which have known properties, such as those in equation (3.3). We refer to any formula giving an (unknown) generalised ζ function explicitly in terms of one or more known ζ functions as a *regularisation formula.* Such a formula fully reveals the analyticity of the generalised ζ function—its poles and residues in particular.

Previous work [3, 9] on regularisation formulae has concentrated on single-sum ζ functions. It was possible to deal with a rather large class of ζ functions by repeated use of the binomial theorem; for example,

$$\begin{aligned}\sum_{m=1}^{\infty} (m^{\alpha} + a)^{-s} &= \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} a^k m^{-\alpha(s+k)} \\ &= \sum_{k=0}^{\infty} \binom{-s}{k} a^k \zeta(\alpha(s+k)) \qquad \alpha > 0.\end{aligned} \tag{3.4}$$

Here it is obvious that the left-hand side converges for $\mathrm{Re}\, \alpha s > 1$—in fact, converges absolutely since all terms are positive—because the higher terms are $\sim m^{-\alpha s}$. The divergence of the series for $\mathrm{Re}\, \alpha s = 1$ is expressed by the $k = 0$ term in the final equality. Moreover, the binomial series representing $(m^{\alpha} + a)^{-s}$ is absolutely convergent for $|a| < 1$, while the sum over m representing $\zeta(\alpha(s+k))$ is absolutely convergent for $\mathrm{Re}\, \alpha s > 1$. Under these circumstances, standard theorems in series analysis (see, e.g., [10]) assure us that the value of the series (3.4) is not affected by commuting Σ_m and Σ_k, and must be correctly given by the final equality.

When one attempts to extend this type of single-sum result to multiple-sum problems, a qualitatively different situation is encountered. True, there are some multiple series for which the binomial theorem can be used essentially as above, e.g.

$$\sum_{n=1}^{\infty} \sum_{m=2}^{\infty} (n^{\alpha}+1/m^{\beta})^{-s}$$

$$= \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} (1/m^{\beta})^{k} n^{-\alpha(s+k)}$$

$$= \sum_{k=0}^{\infty} \binom{-s}{k} [\zeta(\beta k)-1]\zeta(\alpha(s+k)) \qquad \alpha, \beta > 0. \tag{3.5}$$

Here $a<1$ in (3.4) has been replaced by $m^{-\beta}<1$ and the binomial theorem still applies. Physical problems, however, generally involve series of the form

$$\sum_{n,m=1}^{\infty} (n^{\alpha}+m^{\beta})^{-s} \qquad \alpha, \beta > 0. \tag{3.6}$$

Here neither term in the binomial is guaranteed larger than the other and the binomial theorem cannot be used to factorise the n and m sums. Therefore Σ_n and Σ_m cannot be separately evaluated in terms of Riemann ζ functions.

Fortunately, using methods tailored to specific cases, it is possible to obtain regularisation formulae for multiple-sum ζ functions, as we now show.

3.2. *Multiple-sum ζ functions*

Consider the ζ function

$$Z_N^+(s) \equiv \sum_{n_i=1}^{\infty} (n_1+\ldots+n_N)^{-s}. \tag{3.7}$$

Observe that

$$\sum_{n_i=1}^{\infty} f(n_1+\ldots+n_N) = \frac{1}{(N-1)!} \sum_{m=1}^{\infty} (m-1)(m-2)\ldots(m-N+1) f(m) \tag{3.8}$$

for any function $f(x)$, where the coefficient of $f(m)$ is the number of ways the integers n_i can add up to give $m=n_1+\ldots+n_N$. From (3.7) and (3.8) we find the finite regularisation formula

$$Z_N^+(s) = \frac{1}{(N-1)!} \sum_{k=0}^{N-1} C_k^N \zeta(s-k) \tag{3.9}$$

where

$$(m-1)(m-2)\ldots(m-N+1) \equiv \sum_{k=0}^{N-1} C_k^N m^k. \tag{3.10}$$

$Z_N^+(s)$ has simple poles at $s=1,2,\ldots,N$. The first three Z_N^+ are

$$\begin{aligned} Z_2^+(s) &= \zeta(s-1)-\zeta(s) \\ 2Z_3^+(s) &= \zeta(s-2)-3\zeta(s-1)+2\zeta(s) \\ 6Z_4^+(s) &= \zeta(s-3)-6\zeta(s-2)+11\zeta(s-1)-6\zeta(s). \end{aligned} \tag{3.11}$$

The rearrangment of terms leading from (3.7) to (3.9) is allowable because the multiple series (3.7) is manifestly, absolutely convergent for large values of Re s. Thus (see, e.g., [10]) arbitrary rearrangements of terms in (3.7) can have no effect on the value of this series. Its value is correctly given by the right-hand side in (3.9) as long as the left-hand side is convergent. Once (3.9) is established for large Re s, the right-hand side can then be used to continue the left-hand side throughout the s plane.

To make the procedure a little more transparent, consider

$$\begin{aligned} Z_2^+(s) &= \sum_{n,m=1}^{\infty} (n+m)^{-s} \\ &= \frac{1}{2^s} + \frac{2}{3^s} + \frac{3}{4^s} + \frac{4}{5^s} + \ldots \\ &= \sum_{m=1}^{\infty} \frac{m-1}{m^s} = \zeta(s-1) - \zeta(s). \end{aligned}$$

The right-hand side has poles at $s = 1, 2$. The $s = 2$ pole tells us that the double series diverges for Re $s = 2$. For Re $s > 2$ this series is absolutely convergent. But with the help of the right-hand side, one can continue the function defined by the double series to all values of s.

All of the regularisation formulae in this section belong to one of two categories: formulae like (3.4) or (primarily) formulae like (3.9). Both categories are absolutely convergent for suitable s and regrouping of terms can be done freely. The final results obtained can be justified by the same arguments used for (3.4) and (3.9).

Equation (3.8) is quite helpful in that it can be used to find the regularisation formulae of many other ζ functions—whenever the summand depends only on $n_1 + n_2 + \ldots + n_N$. An example is the alternating sign series:

$$\begin{aligned} Z_N^-(s) &\equiv \sum_{n_i=1}^{\infty} (-1)^{n_1+\ldots+n_N+1} (n_1 + \ldots + n_N)^{-s} \\ &= \frac{1}{(N-1)!} \sum_{k=0}^{N-1} C_k^N \eta(s-k). \end{aligned} \tag{3.12}$$

For the most part, alternating sign series can be handled much like their non-alternating sign partners. For this reason, we shall concentrate on the latter in this section.

Consider next the example

$$\begin{aligned} Z_N^\pm(s, a, \alpha) &\equiv \sum_{n_i=1}^{\infty} (\pm)^{n_1+\ldots+n_N+1} ((n_1 + \ldots + n_N)^\alpha + a)^{-s} \\ &= \sum_{k=0}^{\infty} \binom{-s}{k} a^k \frac{1}{(N-1)!} Z_N^\pm(\alpha(s+k)) \end{aligned} \tag{3.13}$$

where $\alpha > 0$. Here the binomial theorem is used exactly as in (3.4). It is worth mentioning that for $\alpha = 1$ the ζ function (3.13) can be expressed as a finite series in terms of the generalised Riemann ζ function $\zeta(s, a)$. For example (see [7] for a different derivation)

$$\begin{aligned} Z_2^+(s, a, 1) &= \sum_{m=0}^{\infty} (m+1-a-1)(m+a)^{-s} + a^{-s} \\ &= \zeta(s-1, a) - (1+a)\zeta(s, a) + a^{-s}. \end{aligned} \tag{3.14}$$

A well known property [5] of $\zeta(s,a)$ are the special values $\zeta(-m,a)=-(m+1)^{-1}B_{m+1}(a)$ in terms of Bernoulli polynomials. More general ζ functions of this type have a similar property [3]. Thus by setting $s=-m$ in (3.4) and (3.13) we obtain polynomials because the binomial coefficient cuts off the sum over k.

Next, observe that

$$\sum_{n_i=1}^{\infty} n_j(n_1+\ldots+n_N)^{-s}=\frac{1}{N}Z_N^{+}(s-1) \qquad j \text{ fixed} \tag{3.15}$$

is independent of the label j and trivial to derive. The same trick can be used to obtain many other formulae, e.g.

$$\sum_{n_i=1}^{\infty} n_j(n_1+\ldots+n_N+a)^{-s}=\frac{1}{N}Z_N^{+}(s-1,a,1)-\frac{a}{N}Z_N^{+}(s,a,1). \tag{3.16}$$

To generalise the preceding discussion, let us consider the series

$$\sum_{n,m=1}^{\infty} n^{\alpha}(n+m)^{-s}=\sum_{k=1}^{\infty}\frac{1}{k^s}g_\alpha(k)$$
$$g_\alpha(k)=1+2^\alpha+\ldots+(k-1)^\alpha \qquad \alpha>0. \tag{3.17}$$

For integral $\alpha=N$ the MacLaurin summation formula [4] can be used to evaluate $g_N(k)$:

$$g_N(k)=\frac{1}{N+1}k^{N+1}+B_1k^N+\tfrac{1}{2}B_2\binom{N}{1}k^{N-1}$$
$$+\tfrac{1}{4}B_4\binom{N}{3}k^{N-3}+\tfrac{1}{6}B_6\binom{N}{5}k^{N-5}+\ldots \tag{3.18}$$

where k^0 and negative powers of k are omitted and the B_n are Bernoulli numbers. Thus

$$\sum_{n,m=1}^{\infty} n^N(n+m)^{-s}$$
$$=\frac{1}{N+1}\zeta(s-N+1)+B_1\zeta(s-N)+\tfrac{1}{2}B_2\binom{N}{1}\zeta(s-N+1)$$
$$+\tfrac{1}{4}B_4\binom{N}{3}\zeta(s-N+3)+\ldots \tag{3.19}$$

where for even N the series terminates at $\zeta(s-1)$, and for odd N at $\zeta(s-2)$, e.g.

$$\sum_{n,m=1}^{\infty} n^2(n+m)^{-s}=\tfrac{1}{6}[2\zeta(s-3)-3\zeta(s-2)+\zeta(s-1)]$$
$$\sum_{n,m=1}^{\infty} n^3(n+m)^{-s}=\tfrac{1}{4}[\zeta(s-4)-2\zeta(s-3)+\zeta(s-2)] \tag{3.20}$$
$$\sum_{n,m=1}^{\infty} n^4(n+m)^{-s}=\tfrac{1}{30}[6\zeta(s-5)-15\zeta(s-4)+10\zeta(s-3)-\zeta(s-1)].$$

Again the rearrangment of terms leading to these regularisation formulae can be justified by the absolute convergence of the left-hand series for sufficiently large Re s. For non-integral α the Euler-MacLaurin formula does not help us evaluate the 'degeneracy' factor $g_\alpha(k)$ in (3.17) and we are unable to give an explicit regularisation formula.

We mention in passing that one can regularise higher sums of the type (3.17) in terms of the series (3.17) itself, e.g.

$$\sum_{n_i=1}^{\infty} n_3^{\alpha}(n_1+n_2+n_3)^{-s} = \sum_{k=1}^{\infty} (k-1) \sum_{n_3=1}^{\infty} n_3^{\alpha}(k+n_3)^{-s}. \tag{3.21}$$

Using manipulations of this nature, one is able to set up a hierarchy of ζ functions, with more complicated ones being regularised in terms of simpler ones, and the entire edifice built upon one's ability to explicitly regularise the simplest ζ function at the bottom of the edifice.

Next consider the series

$$\sum_{n,m=1}^{\infty} n^{\alpha} m^{\beta}(n+m)^{-s} = \sum_{k=1}^{\infty} \frac{1}{k^s} g(k) \tag{3.22}$$

$$g(k) = \sum_{n=1}^{k-1} (k-n)^{\alpha} n^{\beta}. \tag{3.23}$$

For integral $\alpha = N$, $\beta = M$ we can evaluate $g(k)$ by the Euler-MacLaurin formula [4]:

$$\begin{aligned} g(k) = k^{M+N+1} \sum_{r=0}^{N} \binom{N}{r} \frac{(-1)^{N-r}}{M+N+1-r} + \sum_{r=0}^{N} \binom{N}{r} (-1)^{N-r} k^r \\ \times \left\{ \tfrac{1}{2} B_2 \binom{M+N-r}{1} k^{M+N-r-1} + \tfrac{1}{4} B_4 \binom{M+N-r}{3} k^{M+N-r-3} \right. \\ \left. + \tfrac{1}{6} B_6 \binom{M+N-r}{5} k^{M+N-r-5} + \dots \right\} \end{aligned} \tag{3.24}$$

where, in the curly bracket, terms with negative powers of k as well as the k^0 term are discarded. Inserting $g(k)$ in (3.22) we obtain the regularisation formula for this ζ function with $\alpha = N$, $\beta = M$. For non-integral α, β we encounter the same problem as before. Again, higher sums of the same type can be regularised in terms of the ζ function (3.22).

Up to now we have dealt only with series of the form (3.1) with $a_i = 1$. For arbitrary a_i the analysis becomes more complicated. However, the binomial theorem can be used to obtain regularisation formulae. To illustrate the method consider

$$\begin{aligned} \sum_{n,m=1}^{\infty} (n^{\alpha} + am^{\beta})^{-s} \\ = \sum_{n,m=1}^{\infty} [n^{\alpha} + m^{\beta} + (a-1)m^{\beta})^{-s} \qquad \alpha, \beta > 0 \\ = \sum_{k=0}^{\infty} \binom{-s}{k} (a-1)^k \sum_{n,m=1}^{\infty} m^{\beta k}(n^{\alpha} + m^{\beta})^{-s-k}. \end{aligned} \tag{3.25}$$

Here $1 \leqslant a < 2$ and the commutation of sums is justified as before. If one has a regularisation formula for $\Sigma_{n,m}$ in the final line, then (3.25) becomes a regularisation formula for the left-hand side. The device employed in (3.25) can be generally applied to sums of the form (3.1). As this is evident, but one obtains rather lengthy expressions, we refrain from giving more details.

4. Conclusion

The ζ functions considered here are characteristic of multiple harmonic oscillator problems. Field theories defined on the torus T^N (or in an N-dimensional box with periodic or antiperiodic boundary conditions) are characterised by Epstein-type ζ functions [11]

$$\sum (a_1 n_1^2 + \ldots + a_N n_N^2)^{-s}$$

where the power n_i^2 is determined by the quadratic kinetic terms of standard field theory. Epstein ζ functions are much less tractable than the harmonic oscillator ζ functions studied here. Still, it is possible to obtain useful results concerning them. These will be discussed separately along with applications to physical problems.

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